TOPOLOGICAL CONSTRUCTIONS FOR MULTIGRADED SQUAREFREE MODULES

HARA CHARALAMBOUS

Abstract. Let $R = \mathbb{k}[x_1, \ldots, x_n]$ and $M = R^s/I$ a multigraded squarefree module. We discuss the construction of cochain complexes associated to M and we show how to interpret homological invariants of M in terms of topological computations. This is a generalization of the well studied case of squarefree monomial ideals.

1. Introduction

Let $R = \mathbb{k}[x_1, \ldots, x_n]$ be a polynomial ring over the field $\mathbb{k}$ of characteristic 0. For $\alpha = (a_i) \in \mathbb{Z}^n$ we let $x^\alpha = x_1^{a_1} \cdots x_n^{a_n}$ and $R_\alpha = kx^\alpha$. Let $A = (c_{ij}x^{\alpha_{ij}}) \in \mathbb{M}_{s\times l}(R)$: $c_{ij} \in \mathbb{k}$ and $\alpha_{ij} \in \mathbb{N}^n$. We say that A is *multigraded* if each minor of A equals $c_\alpha x^\alpha$ for some $c_\alpha \in \mathbb{k}$. We say that A is of *uniform rank* if all of its minors are nonzero. In particular this implies that $c_{ij} \neq 0$ for all i and j and thus the matrix of coefficients (c_{ij}) is sufficiently generic. In addition we say that $\alpha \in \mathbb{N}^n$ is *squarefree* if $\alpha \in \{0, 1\}^n$ and a collection of vectors is *squarefree* if each vector is squarefree.

We recall that M is a multigraded R-module if $M = \oplus_{\alpha\in\mathbb{N}^n} M_\alpha$ where M_α are subgroups of M and $R_{\alpha_1} M_{\alpha_2} \subset M_{\alpha_1+\alpha_2}$ whenever $\alpha_i \in \mathbb{N}^n$. Moreover $w \in M$ is a multigraded element of M of multidegree β if $w \in M_\beta$, and in this case we write $\deg w = \beta$. Let M be a multigraded finitely generated R-module. M has a minimal multigraded presentation $\phi : R^l \longrightarrow R^s \longrightarrow M \longrightarrow 0$ where for a choice of multigraded generators for R^s and R^l, ϕ is represented by A_M, a multigraded *presentation matrix* of M. We note that the data consisting of the multidegrees of the generators of R^s and R^l and the matrix of coefficients (c_{ij}) describes a *monomial* matrix as in [MiSt05]. In particular whenever $c_{ij} \neq 0$, $\alpha_{ij} =$ (degree of column j)- (degree of row i). We pay special attention to this set of equations. Let $A = (c_{ij}x^{\alpha_{ij}}) \in \mathbb{M}_{s\times l}(R)$, $c_{ij} \in \mathbb{k}$, $\alpha_{ij} \in \mathbb{N}^n$. Whenever $c_{ij} \neq 0$ we consider the equation $\gamma_j - \beta_i = \alpha_{ij}$ with unknowns γ_j, β_i. We assemble these equations to a system E_A of at most $s \cdot l$ equations and $s + l$ unknowns. The fact that A is multigraded reflects the consistency of E_A. For any particular solution $T = (\gamma_1, \ldots, \gamma_l, \beta_1, \ldots, \beta_s)$ of E_A, γ_j gives the degree of the j^{th} column of A and β_i the degree of the i^{th} row of A. Moreover for any such solution T we let F_1, F_0 be the free multigraded modules with bases $B_1 = \{w_j : j \in [l],\ \deg w_j = \gamma_j\}$ and $B_0 = \{v_i : i \in [s],\ \deg v_i = \beta_i\}$ respectively, and let $\phi_T : F_1 \longrightarrow F_0$, $\phi_T(w_j) = \sum_i c_{ij}x^{\alpha_{ij}} v_i$. The module $M_T = \operatorname{Coker} \phi_T$ is multigraded and has A as a presentation matrix. We will occasionally write M_A for M_T.

The multigraded module M is called *squarefree* if the function $M_\alpha \to M_{\alpha+\beta}$: $y \mapsto x^\beta y$ is a bijection whenever $\operatorname{supp}(\beta) \subset \operatorname{supp}(\alpha)$, see [Ya00]. In some sense it suffices to study squarefree multigraded modules: as is shown in [BrHe95] or [ChDe01], if M is any multigraded module then there is a a squarefree multigraded

module L with the same homological properties as M. In this paper we show that the multigraded matrix A is the presentation matrix of a squarefree module if and only if there exists a squarefree solution to E_A. Such a matrix is called *squarefree*. It follows that all nonzero entries of a squarefree matrix have squarefree degrees.

When $M = R/I$ and I is a monomial squarefree ideal, the simplicial complex $\Delta_I = \{\{i_1, \ldots, i_t\} \subset \{1, \ldots, n\} : \ x_{i_1} \cdots x_{i_t} \notin I\}$, is well studied and properties of M translate to combinatorial properties of Δ_I. We generalize the above to any multigraded squarefree module M. For this we use a sequence of monomial squarefree ideals that are associated to the presentation of M. When the multigraded presentation matrix A_M is of uniform rank such a set of ideals is explicitly computed in terms of the rows of A_M. A preliminary version of these results (without proofs) has appeared in [Ch06].

We describe the main results in each section. We show that if the multigraded squarefree module M has a minimal multigraded presentation $\phi : R^l \longrightarrow R^s \longrightarrow M \longrightarrow 0$ then there are s squarefree monomial ideals $I_1, \ldots, I_s$ that determine a multigraded $\Bbbk$-basis M, Theorem 2.3 and Corollary 2.4. This translates as follows in Gröbner basis language: consider a term order on R^s based on any monomial order on R and an ordering of the multigraded bases elements $v_i, i \in [s]$ of R^s; the initial module of the image of ϕ is a direct sum $I_1 v_1 \oplus \cdots \oplus I_s v_s$. It follows that if A is any multigraded squarefree matrix then there is a multigraded squarefree module M with presentation matrix A, Corollary 2.5. We study the annihilator ideal of a multigraded squarefree module M_A when A is an $s \times l$ matrix of uniform rank: when $s > l$ we show that $\operatorname{ann}(M) = 0$ while when $s \leq l$ we show that $\operatorname{ann}(M)$ equals the radical of the ideal generated by the $s \times s$ minors of A, Theorem 2.11.

In the next section we study in more detail the case of a squarefree multigraded module whose presentation matrix A_M is of uniform rank. In this case we show that the squarefree monomial ideals that determine a basis of M are generated by least common multiples of monomials in the appropriate rows of A_M, Theorem 3.4. Their intersection equals $\operatorname{ann}(M)$. Thus the dimension of M can be computed based on these ideals.

In the last two sections of this paper we assume M to be a squarefree multigraded module. In section 4, for each $\alpha \in \mathbb{Z}^n$ we construct a cochain complex and use it to compute the α-graded betti numbers of M. In the last section for each $\alpha \in \mathbb{Z}^n$ we construct a complex to calculate the α-graded piece of the local cohomology of M.

We refer to [Ei97], [BrHe98] and [MiSt05] for undefined terms and notation. We also want to thank the referee for suggesting the more general version of Theorem 2.3 and Corollary 2.4 and the generalization of the last two sections.

2. Squarefree multigraded matrices

For $\alpha = (a_i) \in \mathbb{Z}^n$, we let $\operatorname{supp}(\alpha) = \{i : \ a_i \neq 0\} \subset [n]$. When $\alpha \in \mathbb{N}^n$ we write σ_α for $\operatorname{supp}(\alpha)$ and denote by q_α the squarefree vector so that $\sigma_\alpha = \sigma_{q_\alpha}$. If $t \in \mathbb{N}$ by $[t]$ we denote the set $\{1, \ldots, t\}$.

Definition 2.1. A multigraded $s \times l$ matrix $A = (c_{ij} x^{\alpha_{ij}})$ is called *squarefree* if the system E_A has a squarefree solution T.

Remark 2.2. Let M be a squarefree multigraded module and let ϕ be a minimal multigraded presentation ϕ of M. Since the kernel of ϕ is a squarefree module

[Ya00] it follows that the minimal multigraded generating sets of M and $\ker\phi$ have squarefree degrees. Thus A_M is squarefree.

Let $T = (\gamma_1, \dots, \gamma_l, \beta_1, \dots, \beta_s)$ be a squarefree solution of E_A, $\phi_T : F_1 \longrightarrow F_0$. For any monomial order on R and an ordering of the basis elements of F_0, we let $>$ be the following monomial order on F_0: $uv_i > u'v_j$ if $v_i > v_j$ or $i = j$ and $u > u'$. We denote by $\operatorname{in}(\operatorname{Im}\phi_T)$ the initial module of $\operatorname{Im}(\phi_T)$ with respect to $>$.

Theorem 2.3. *Let $A = (c_{ij}x^{\alpha_{ij}})$ be a multigraded squarefree $s \times l$ matrix, T a squarefree solution of E_A and $>$ a term order on F_0 as above. There exist squarefree monomial ideals $I_1, \dots, I_s$ of R such that*

$$\operatorname{in}(\operatorname{Im}\phi_T) = I_1 v_1 \oplus \cdots \oplus I_s v_s$$

Proof. Without loss of generality we can assume that $v_s > \ldots > v_1$. Let $f \in \operatorname{Im}\phi$, f multigraded, $\deg f = \alpha$, $\operatorname{in}(f) = x^{\alpha_i} v_i$. Thus for $j \in [s]$ and $t \in [l]$, there exist $c_j \in \Bbbk$ and $r_t \in R_{\alpha-\gamma_t}$ such that

$$f = \sum_{j \le i} c_j x^{\alpha_j} v_j = \sum_t r_t \phi_T(w_t) \ .$$

For $t \in [l]$, let $d_t = \deg r_t = \alpha - \gamma_t$. Since $\gamma_t \in \{0,1\}^n$, it follows that whenever $r_t \neq 0$, $d_t - (\alpha - q_\alpha) = q_\alpha - \gamma_t \in \{0,1\}^n$. Moreover since $\alpha_j = \alpha - \beta_j$ and $\beta_j \in \{0,1\}^n$, it follows that whenever $c_j \neq 0$, ${\alpha_j}' = \alpha_j - (\alpha - q_\alpha) \in \{0,1\}^n$. Thus ${r_t}' = r_t / x^{\alpha - q_a} \in R$ for $t \in [l]$ and

$$f' = \sum_t {r_t}' \phi_T(w_t) \in \operatorname{Im}(\phi_T) \ .$$

Since $\operatorname{in}(f') = c_i x^{{\alpha_i}'} v_i$ and $\alpha_i - {\alpha_i}' \in \mathbb{N}^n$ we are done. ☐

We let $M_T = \operatorname{Coker}\phi_T$ and write $\overline{g}$ for $g + \operatorname{Im}\phi_T$. We note the following:

Corollary 2.4. *Let $A = (c_{ij}x^{\alpha_{ij}})$ be a multigraded squarefree $s \times l$ matrix, T a squarefree solution of E_A, $M_T = \operatorname{Coker}\phi_T$. There exist simplicial complexes $\Delta_1, \dots, \Delta_s$ such that*

1. *the set $B(M_T) = \{\overline{x^\beta v_i} : \ \sigma_\beta \in \Delta_i, \ i \in [s]\}$ is a $\Bbbk$-basis of M_T,*
2. *if $\alpha \in \mathbb{N}^n$ and $\sigma(\alpha) \notin \Delta_i$, then for each $j < i$ there are unique $r_{i,j,\alpha} \in \Bbbk$ such that $r_{i,j,\alpha} = 0$ when $\sigma(\alpha + \beta_i - \beta_j) \notin \Delta_j$ and*

$$\overline{x^\alpha v_i} = \sum_{j<i} r_{i,j,\alpha} \ \overline{x^{\alpha+\beta_i-\beta_j} v_j} \ .$$

Proof. We let $I_1, \dots, I_s$ be the ideals of of Theorem 2.3 with $v_s > \ldots > v_1$ and we let Δ_i be the simplicial complex Δ_{I_i}. The first part follows from Macaulay's Lemma, see for example [Ei97, Theorem 15.3]. For the second part we note that if $\sigma(\alpha) \notin \Delta_i$ and $x^\alpha \in I_i$ then there is an $f_\alpha \in \operatorname{Im}(\phi_T)$ such that $\operatorname{in}(f_\alpha) = x^\alpha v_i$ and

$$x^\alpha v_i - f_\alpha = \sum_{j<i} c_j x^{\alpha_j} v_j \ .$$

In particular $\alpha_j = \alpha + \beta_i - \beta_j$. A repeated application of this remark gives the desired result. ☐

The next corollary justifies the definition of a squarefree matrix.

Corollary 2.5. *Let A be a multigraded squarefree matrix. Then A is the presentation matrix of a multigraded squarefree module M.*

Proof. Let T be a squarefree solution of E_A, $M_T = \operatorname{Coker}\phi_T$. By Corollary 2.4, $(M_T)_\alpha \cong (M_T)_{q_\alpha}$ and M_T is squarefree. $\square$

The *join* of $\alpha_1, \ldots, \alpha_t \in \mathbb{N}^n$ denoted $\operatorname{join}(\alpha_1, \ldots, \alpha_t)$ is the vector with components the maximum of the corresponding components of the $\alpha_1, \ldots, \alpha_t$. We will need the following lemma.

Lemma 2.6. *Let $A = (c_{ij}x^{\alpha_{ij}})$ be a multigraded $s \times l$ of uniform rank where α_{ij} are squarefree, and let $t, q \in [l]$ and $f, i_1, \ldots, i_r \in [s]$. Then*

$$\frac{\operatorname{lcm}(x^{\alpha_{tf}}, x^{\alpha_{ti_1}}, \cdots, x^{\alpha_{ti_r}})}{x^{\alpha_{tf}}} = \frac{\operatorname{lcm}(x^{\alpha_{qf}}, x^{\alpha_{qi_1}}, \cdots, x^{\alpha_{qi_r}})}{x^{\alpha_{qf}}} \ ,$$

or equivalently

$$\operatorname{join}(\alpha_{tf}, \alpha_{ti_1}, \ldots, \alpha_{ti_r}) - \alpha_{tf} = \operatorname{join}(\alpha_{qf}, \alpha_{qi_1}, \cdots, \alpha_{qi_r}) - \alpha_{qf} \ .$$

Moreover if $f, j_1, \ldots, j_r \in [s]$ and $q, t \in [l]$ then

$$\operatorname{join}(\alpha_{ft}, \alpha_{j_1t}, \ldots, \alpha_{j_rt}) - \alpha_{ft} = \operatorname{join}(\alpha_{fq}, \alpha_{j_1q}, \cdots, \alpha_{j_rq}) - \alpha_{fq}$$

Proof. We will show the first equality. We note that the last equality is a consequence of the first, since A^T is multigraded. The expressions on either side of the equation are squarefree monomials. Suppose that the variable x_j divides the left hand side expression. This implies that x_j does not divide $x^{\alpha_{tf}}$ and x_j divides $x^{\alpha_{ti_h}}$ for some i_h, where $h = 1, \ldots, r$. Since $c_{tf}c_{qi_h}x^{\alpha_{tf}+\alpha_{qi_h}} - c_{ti_h}c_{qf}x^{\alpha_{ti_h}+\alpha_{qf}}$ is a minor of A and A is multigraded it follows that x_j divides $x^{\alpha_{qi_h}}$ and x_j does not divide $x^{\alpha_{qf}}$. Thus x_j divides the right hand side of the equation. $\square$

We can now prove the following:

Proposition 2.7. *Any multigraded matrix of uniform rank whose entries have squarefree degrees is squarefree.*

Proof. Let $A = (c_{ij}x^{\alpha_{ij}})$ be a multigraded of uniform rank, where $\alpha_{ij} \in \{0,1\}^n$ for $i \in [s]$, $j \in [l]$. By Lemma 2.6 a solution to E_A is given by $\gamma_j = \operatorname{join}(\alpha_{ij} : \ i \in [s])$ for $j \in [l]$, and $\beta_i = \gamma_1 - \alpha_{i1}$ for $i \in [s]$. $\square$

Remarks 2.8.

- We note that when A is of uniform rank then E_A has one degree of freedom.
- When $A = (c_{ij}x^{\alpha_{ij}})$ is not of uniform rank then E_A might not have squarefree solutions even when $\alpha_{ij} \in \{0,1\}^n$ for $c_{ij} \neq 0$. For example, let $R = \Bbbk[x, y]$ and
$$A = \begin{bmatrix} x & y \\ 0 & x \end{bmatrix}.$$
The general solution of E_A consists of $\gamma_1 = (2+t, s)$, $\gamma_2 = (1+t, 1+s)$, $\beta_1 = (1+t, s)$, $\beta_2 = (t, s+1)$.
- When $A = (c_{ij}x^{\alpha_{ij}})$ is of uniform rank and $s = 1$ then $B_0 = \{v_1 : \beta_1 = \deg v_1 = 0\}$, $B_1 = \{w_j : \gamma_j = \deg w_j = \alpha_{1j}\}$ and $M_T = R/I$ where $I = \langle x^{a_{11}}, \ldots, x^{a_{1l}} \rangle$.

We will close this section by examining the annihilator of a multigraded squarefree module M in terms of its multigraded presentation matrix A. It is easy to see that $\operatorname{ann}(M)$ is generated by monomials. If A is $s \times l$ and $l \geq s$ we denote by $\operatorname{Fitt}_0(M)$ the ideal of the $s \times s$ minors of A. We note that the generators of $\sqrt{\operatorname{Fitt}_0(M)}$ are least common multiples of the entries in a diagonal of

A of length s. It is well known, see [Ei97], that $\mathrm{Fitt}_0(M) \subset \mathrm{ann}(M)$ and that $\mathrm{ann}(M)^s \subset \mathrm{Fitt}_0(M)$. In the case where $\mathrm{ann}(M)$ is a monomial ideal it follows that $\mathrm{ann}(M) \subseteq \sqrt{\mathrm{Fitt}_0(M)}$. For what follows we write $\mathrm{diag}(x^\alpha, s)$ for the $s \times s$ identity matrix times x^α. We will use the following lemma which characterizes the elements of $\mathrm{ann}(M)$.

Lemma 2.9. *Let A be multigraded squarefree $s \times l$ matrix and let M be a module with presentation matrix A. The annihilator of M consists of all monomials x^α such that the linear system*

$$AX = \mathrm{diag}(x^\alpha, s)$$

has a solution in $\mathbb{M}_{l\times s}(R)$.

Proof. Let $\phi : F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$ be such that for a basis $\{v_i : \ i \in [s]\}$ of F_0 and a basis $\{w_j : \ j \in [l]\}$ of F_1, ϕ is represented by $A = (c_{ij}x^{\alpha_{ij}})$. Since $M = F_0/\operatorname{Im}(\phi)$, it follows that $x^\alpha \in \mathrm{ann}(M)$ if and only if $x^\alpha v_i \in \operatorname{Im}(\phi)$. Thus $x^\alpha \in \mathrm{ann}(M)$ if and only if for any $i \in [s]$ there exist $r_{i1}, \ldots, r_{il}$ such that

$$x^\alpha v_i = \sum_{j=1}^{l} r_{ij} \sum_{t=1}^{s} (c_{tj}x^\alpha_{tj})v_t. \tag{2.10}$$

We let C_i be the ith column of $\mathrm{diag}(x^\alpha, s)$. Since $\{v_i : \ i \in [s]\}$ is a basis for F_0 the system 2.10 is consistent if and only if $AX = C_i$ is consistent for each $i \in [s]$. $\square$

If $K_1 \subset [s]$ and $K_2 \subset [l]$ we denote by $A[K_2, K_1]$ the submatrix of A consisting of the rows indexed by K_1 and columns indexed by K_2. The next proposition computes the annihilator of the squarefree module M when the presentation matrix is of uniform rank.

Proposition 2.11. *Let A be a multigraded squarefree $s \times l$ matrix of uniform rank. Suppose that A is the presentation matrix of M. If $l < s$ then* $\mathrm{ann}(M) = 0$. *Otherwise* $\mathrm{ann}(M) = \sqrt{\mathrm{Fitt}_0(M)}$.

Proof. If $l < s$ and $x^\alpha \in \mathrm{ann}(M)$ then by Lemma 2.9 assume that Z is such that

$$AZ = \mathrm{diag}(x^\alpha, s) \ .$$

Without loss of generality we can assume that the first column Z_1 of Z is nonzero. It follows that $A[[l], \{2, \ldots, l+1\}] \cdot Z_1 = 0$ and thus $\det A[[l], \{2, \ldots, l+1\}] = 0$, a contradiction since A is of uniform rank.

Suppose now that $l \geq s$. We will show that $\sqrt{\mathrm{Fitt}_0(M)} \subset \mathrm{ann}(M)$. Let $I = \sqrt{\mathrm{Fitt}_0(M)}$ and let x^α be a minimal generator of I. It follows that there is a set $K = \{j_1, \ldots, j_s : \ j_1 < j_2 < \ldots < j_s\} \subset [l]$ such that $x^\alpha = \det A[K, [s]]$. Since A is multigraded it follows that for any $i \in [s]$, $\det A[K, [s]] = c_{K,t}x^{\alpha_{ij_t}} \det A[K \setminus j_t, \{1, \ldots, \hat{i}, \ldots, s\}]$, where $c_{K,t} \in \Bbbk$. For $i \in [s]$, we let C_i be as in the proof of Lemma 2.9. By Cramer's rule it follows that the system $A[K, [s]]X = C_i$ has a solution $Z = (z_{j1}) \in \mathbb{M}_{s\times 1}(R)$ with entries in R. We extend Z to a solution $Y \in \mathbb{M}_{l\times 1}(R)$ for the system $AX = C_i$ by setting $y_{j1} = z_{j1}$ when $j \in K$ and letting $y_{j1} = 0$ if $j \notin K$. Thus by Lemma 2.9, $x^\alpha \in \mathrm{ann}(M)$ and $I \subset \mathrm{ann}(M)$ as desired. $\square$

3. Squarefree matrices of uniform rank

In the previous section for any multigraded squarefree matrix A and any solution T of E_A we proved the existence of a sequence of squarefree monomial ideals that provide a basis for M_T. In this section we compute these ideals when A is squarefree of uniform rank.

Definition 3.1. Let $A = (c_{ij}x^{\alpha_{ij}})$ be a multigraded squarefree matrix of uniform rank with $l \geq s$. For $i \in [s]$ we let

$$I_i := \langle\ \mathrm{lcm}(x^{\alpha_{ij_1}}, \ldots, x^{\alpha_{ij_{s-i+1}}}) :\ 1 \leq j_1 < \cdots < j_{s-i+1} \leq l\ \rangle\ .$$

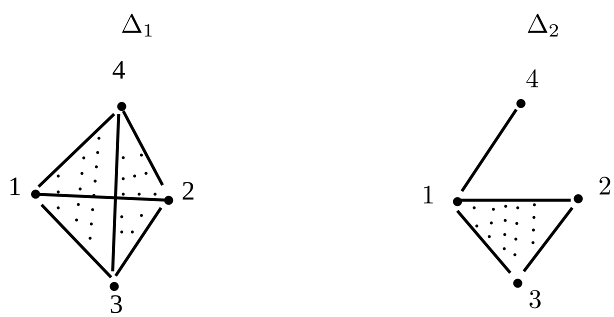
We denote by Δ_i the simplicial complex Δ_{I_i}.

For the rest of this section we assume that A is of uniform rank as above.

Example 3.2. Let $R = \Bbbk[x, y, z, w]$ and

$$A = \begin{bmatrix} xy & xz \\ wy & 2wz \end{bmatrix}.$$

A is the matrix of the R-module homomorphism $\phi_1 : F_1 \longrightarrow F_0$ where $F_1 = Rw_1 \oplus Rw_2$, $F_0 = Rv_1 \oplus Rv_2$, $\deg w_1 = \gamma_1 = (1, 1, 0, 1)$, $\deg w_2 = \gamma_2 = (1, 0, 1, 1)$, $\deg v_1 = \beta_1 = (0, 0, 0, 1)$, $\deg v_2 = \beta_2 = (1, 0, 0, 0)$, $M_A = \operatorname{Coker} \phi_1$. Here $I_1 = (xyz)$ and $I_2 = (wy, wz)$. Below we graph the simplicial complexes Δ_1, Δ_2.



We note that when $s = 1$ then $I_1 = \langle x^{a_{11}}, \ldots, x^{a_{1l}} \rangle$ and the unique multigraded $\Bbbk$-basis of R/I_1 is the set $\{x^\beta : \sigma_\beta \notin \Delta_1\}$. We recall that if $T = (\gamma_j : j \in [l], \beta_i : i \in [s])$ is a solution of E_A then F_1, F_0 are the free multigraded modules with bases $B_1 = \{w_j : j \in [l],\ \deg w_j = \gamma_j\}$, $B_0 = \{v_i : i \in [s],\ \deg v_i = \beta_i\}$ respectively, $\phi_T : F_1 \longrightarrow F_0$, $\phi_T(w_j) = \sum_i c_{ij} x^{\alpha_{ij}} v_i$ and $M_A = \operatorname{Coker} \phi_T$.

Definition 3.3. We define

$$\mathcal{B}_{jA} := \{\overline{x^\beta v_j} :\ x^\beta \notin I_j\} = \{\overline{x^\beta v_j} :\ \sigma_\beta \in \Delta_j\}\ ,$$

$$\mathcal{B}_A := \bigcup_{j=1}^{s} \mathcal{B}_{jA}$$

The elements of $\mathcal{B}_A$ are multigraded and $\deg \overline{x^\beta v_j} = \beta + \beta_j$.

Theorem 3.4. *Let $A = (c_{ij}x^{\alpha_{ij}})$ be an $s \times l$ multigraded squarefree matrix of uniform rank. The set $\mathcal{B}_A$ is a multigraded $\Bbbk$-basis for M_A.*

Proof. We use induction on s. For $s = 1$ the theorem is clear. Let $s > 1$. First we show that the elements of $\mathcal{B}_A$ are linearly independent. Suppose that

$$\sum_{j\in[s]}\sum_{i\in K_j} k_{ji}x^{\beta_{ji}}v_j = \sum_{f=1}^{l}\sum_{t=1}^{s} r_f(c_{tf}x^{\alpha_{tf}})v_t$$

is a dependence relation; K_j is a finite index set for each $j \in [s]$, $k_{ji} \in \Bbbk$, and $\overline{x^{\beta_{ji}}v_j} \in \mathcal{B}_{jA}$. It follows that

$$\sum_{i\in K_1} k_{1i}x^{\beta_{1i}}v_1 = \sum_{f=1}^{l} r_f c_{1f}x^{\alpha_{1f}}v_1 \tag{3.5}$$

while

$$\sum_{t=2}^{s}\sum_{i\in K_t} k_{ti}x^{\beta_{ji}}v_t = \sum_{f=1}^{l}\sum_{t=2}^{s} r_f c_{tf}x^{\alpha_{tf}}v_t. \tag{3.6}$$

Let ${F_0}' = Rv_2 \oplus \cdots \oplus Rv_s$, $\phi' : F_1 \longrightarrow {F_0}'$ be the R-homomorphism whose matrix with respect to the bases $\{w_i : i \in [l]\}$ of F_1 and $\{v_i : i \in \{2,\ldots,s\}\}$ of ${F_0}'$ is the submatrix $A' = A[\{1,\ldots,l\},\{2,\ldots,s\}]$ of A. Thus $M_{A'} = {F_0}'/\operatorname{Im}\phi'$. We note that for $j = 1,\ldots,s-1$ there is a one-to-one correspondence between the elements of $\mathcal{B}_{jA'}$ and $\mathcal{B}_{j+1A}$. Thus the expression of Equation 3.6 translates to a dependence relation for the elements $\mathcal{B}_{A'}$. According to the induction hypothesis this implies that $k_{ti} = 0$ for $t \geq 2$ and $i \in K_t$. Thus it remains to show that $k_{1i} = 0$. We examine the coefficients r_f that appear on the right hand side of Equations 3.5 and 3.6. Since the sum on the left hand side of Equation 3.6 is zero, it follows that $r_1w_1 + \cdots + r_lw_l \in \ker\phi_1'$. According to the description of the free resolution of $M_{A'}$ see [ChTc03] it follows that for $1 \leq j \leq l$,

$$r_j \in \langle\ \frac{\operatorname{lcm}(x^{\alpha_{2j}}, x^{\alpha_{2i_1}},\cdots,x^{\alpha_{2i_{s-1}}})}{x^{\alpha_{2j}}} :\ 1 \leq i_1 < \cdots < i_{s-1} \leq l, i_t \neq j\ \rangle.$$

By Lemma 2.6,

$$\frac{\operatorname{lcm}(x^{\alpha_{2j}}, x^{\alpha_{2i_1}},\cdots,x^{\alpha_{2i_{s-1}}})}{x^{\alpha_{2j}}} = \frac{\operatorname{lcm}(x^{\alpha_{1j}}, x^{\alpha_{1i_1}},\cdots,x^{\alpha_{1i_{s-1}}})}{x^{\alpha_{1j}}}$$

and $r_jx^{\alpha_{1j}} \in I_1$, a contradiction.

Next we show that $\mathcal{B}_A$ generates M_A. More precisely we will show that if $x^\alpha \in I_i$ then $\overline{x^\alpha v_i}$ can be written as a $\Bbbk$-linear combination of elements of $\mathcal{B}_{1A} \cup \ldots \cup \mathcal{B}_{i-1A}$ of degree $\alpha + \beta_i$. We first show this for the elements of I_1. Let x^α be the least common multiple of the entries in the first row corresponding to the columns indexed by the set $K \subset [l]$, where $|K| = s$. Since A is multigraded $\det A[K,[s]]$ divides $x^\alpha \det A[K \setminus f, \{2,\ldots,s\}]$ for any $f \in K$. By Cramer's rule it follows that there is a matrix $Z = (z_i) \in \mathbb{M}_{s\times 1}(R)$ such that

$$A[K,[s]]Z = \begin{bmatrix} x^\alpha & 0 & \cdots & 0 \end{bmatrix}^T .$$

It follows that if $K = \{i_1,\ldots,i_s\}$ then $x^\alpha v_1 = \phi_1(z_1w_{i_1} + \cdots + z_sw_{i_s})$. Therefore $\overline{x^\alpha v_1} = 0$. We now assume that the statement holds for $j < t$. Let $K = \{i_1,\ldots,i_{s-t+1}\}$ and $x^\alpha \in I_t$ be equal to $\operatorname{lcm}(x^{a_{ti_f}} : i_f \in K)$. Let $Z = (z_f) \in \mathbb{M}_{s-t+1\times 1}(R)$ be such that

$$A[K,\{t,\ldots,s\}]Z = \begin{bmatrix} x^\alpha & 0 & \cdots & 0 \end{bmatrix}^T .$$

For $w = z_1 w_{i_1} + \cdots + z_{s-t+1} w_{i_{s-t+1}}$ we have:

$$\phi_1(w) = z_1(\sum_{j<t} c_{ji_1} x^{\alpha_{ji_1}} v_j) + \cdots + z_{s-t+1}(\sum_{j<t} c_{ji_{s-t+1}} x^{\alpha_{ji_{s-t+1}}} v_j) + x^\alpha v_t$$

$$= \sum_{j<t} (\sum_{r=1}^{r=s-t+1} z_r c_{ji_r} x^{\alpha_{ji_r}} v_j) + x^\alpha v_t.$$

Therefore

$$\overline{x^\alpha v_t} = -\sum_{j<t} (\sum_{r=1}^{r=s-t+1} c_{ji_r} \overline{z_r x^{\alpha_{ji_r}} v_j})$$

and we are done by the induction hypothesis as applied to each of the summands $\overline{z_r x^{\alpha_{ji_r}} v_j}$. □

The next corollary is immediate and we omit its proof.

Corollary 3.7. *Let A be an $s \times l$ multigraded squarefree matrix of uniform rank, $>$ be a term order on F_0 based on a monomial order of R with $v_s > \cdots > v_1$. Then*

$$\text{in}(\text{Im}(\phi_1)) = I_1 v_1 \oplus \cdots \oplus I_s v_s \ .$$

Example 3.8. Let A be the matrix of Example 3.2 and $\alpha = (0,1,0,1)$. Then $\overline{wy\ v_2} = -\overline{xy\ v_1}$ and $r_{2,1,\alpha} = -1$. When $\alpha = (0,0,1,1)$ then $\overline{wz\ v_2} = -1/2\overline{xz\ v_1}$ and $r_{2,1,\alpha} = -1/2$. Note that $xyzv_1 = \phi_1(2z\ w_1 - y\ w_2)$ and $\overline{xyzv_1} = 0$.

A different ordering of the basis elements v_i one would modify Definition 3.1 to get a different set of ideals and a potentially different $\Bbbk$-multigraded basis of M_A. For example if $v_1 > \cdots > v_s$ then the i^{th} ideal should be generated by all least common multiples of i monomials in the i^{th} row of A. Next we describe the annihilator of M in terms of the ideals I_i.

Proposition 3.9. *Let $A = (c_{ij} x^{\alpha_{ij}})$ be an $s \times l$ multigraded squarefree matrix of uniform rank, $l \geq s$. Then*

$$\text{ann}(M_A) = I_1 \cap \cdots \cap I_s \ .$$

Proof. We use Proposition 2.11. We first show that the intersection $I_1 \cap \cdots \cap I_s$ is contained in $\text{ann}(M_A)$. Let $x^{\alpha_i} \in I_i$ for each $i \in [s]$: x^{α_i} determines a (not necessarily unique) subset $L_i \subset [l]$ of cardinality $s-i+1$ such that $x^{\alpha_i} = \text{lcm}(x^{\alpha_{it}} : t \in L_i)$. It follows that there is a set $K = \{i_1, \ldots, i_s\} \subset [l]$ of cardinality s such that $i_t \in L_t$. Since A is multigraded it follows that $\det A[K,[s]]$ divides $\text{lcm}(\alpha_1, \ldots, \alpha_s)$.

For the reverse containment, let x^q be a generator of $\text{ann}(M_A)$: thus there is an ordered set $K = \{j_1, \ldots, j_s\}$ such that $cx^\alpha = \det A[K,[s]]$ and $q = q_\alpha$. Let $x^{\alpha_i} = \text{lcm}(x^{\alpha_{ij_t}} : \ t = i, \ldots, s)$. It is clear that $x^{\alpha_i} \in I_i$. Since A is multigraded it follows that $x^{q_\alpha} = \text{lcm}(x^{\alpha_1}, \ldots, x^{\alpha_s})$. □

The following is now immediate:

Corollary 3.10. *Let A be as above. The dimension of M_A is equal to the least of the codimensions of the ideals I_j.*

4. Betti numbers of M

Let $A = (c_{ij}x^{\alpha_{ij}})$ be an $s \times l$ multigraded squarefree matrix, T a squarefree solution of E_A, $M_A = \operatorname{Coker}\phi_T$. We consider the ideals $I_1, \dots, I_s$ of Theorem 2.3 with respect to a term order induced by $v_s > \dots > v_1$. We start by assembling some notation.

Notation 4.1.

- For $i \in [s]$, we let $\beta_i = \deg(v_i)$. For $\alpha \in \mathbb{Z}^n$ we let $\alpha_j = \alpha - \beta_j$.
- Let $L \subset [n]$. If $t \in L$ then we write $L\hat{t}$ for the set $L \setminus \{t\}$. If $\sigma \subset L$ we write $L\hat{\sigma}$ for the set $L \setminus \sigma$.
- By Lt we denote the set $L \cup \{t\}$, by $L\sigma$ the set $L \cup \sigma$.
- Let $L \subset [n]$. Then $\underline{L} = (d_i)$ where $d_i = 1$ if $i \in L$ and 0 otherwise.
- Let $L = \{i_1, \dots, i_t\}$ where $1 \le i_1 < \dots < i_t \le n$. For $r \in [t]$ we let $\operatorname{sgn}[i_r, L] = (-1)^{r+1}$. For $W \subset L$ we let
$$\operatorname{sgn}[W, L] := \prod_{w \in W} \operatorname{sgn}[w, L].$$
- Let $(K_\bullet, \theta_\bullet)$ be the Koszul complex on the variables $x_1, \dots, x_n$. We denote the multigraded generators of K_j by e_L where $L = \{i_1, \dots, i_j\}$ and $1 \le i_1 < \dots < i_j \le n$ and let $\deg e_L = \underline{L}$. Here
$$\theta(e_L) = \sum_{t \in L} \operatorname{sgn}[t, L] x_t e_{L\hat{t}} \ .$$
- Let Δ be a simplicial complex, $\tau \in \Delta$ and V the vertex set of Δ. We partition $V \setminus \tau$ into two sets: $V_{\tau,\Delta,1} = \{t \notin \tau : \ \tau t \in \Delta\}(=\operatorname{link}_\Delta \tau)$ and $V_{\tau,\Delta,2} = \{t \notin \tau : \ \tau t \notin \Delta\}$.
- Let Δ be a simplicial complex. We let $C^j(\Delta)$ be the $\Bbbk$-vector space with bases elements τ^* where $\tau \in \Delta$ and $|\tau| = j+1$. We let $(C^\bullet(\Delta), d)$ be the augmented cochain complex
$$C^\bullet(\Delta): \quad 0 \longrightarrow C^{-1}(\Delta) \longrightarrow C^0(\Delta) \xrightarrow{d^0} \cdots \longrightarrow C^{n-1}(\Delta) \longrightarrow 0$$
where
$$d^j(\tau^*) = \sum_{t \in V_{\tau,\Delta,1}} \operatorname{sgn}[t, \tau t] \ (\tau t)^*.$$
We let $\widetilde{H}^i(\Delta) = H^i(C^\bullet(\Delta)$.
- For $\alpha \in \mathbb{Z}^n$, we write $\alpha = \alpha^+ - \alpha^-$ where $\alpha^+, \alpha^- \in \mathbb{N}^n$ and $\operatorname{supp}(\alpha^+) \cap \operatorname{supp}(\alpha^-) = \emptyset$.
- Let Δ be a simplicial complex and $\alpha \in \mathbb{N}^n$. We let
$$\Delta_\alpha = \{ \ \sigma \subset \sigma_\alpha : \ \sigma \cup \sigma_{\alpha - q_\alpha} \in \Delta \ \} \ .$$
If $\alpha \in \mathbb{Z}^n \setminus \mathbb{N}^n$ we let $\Delta_\alpha = \{\}$.
- We let $\Delta_{j,\alpha}(A)$ or $\Delta_{j,\alpha}$ for short to be the simplicial complex $(\Delta_{I_j})_{\alpha_j}$. Thus
$$\Delta_{j,\alpha}(A) = \{ \ \sigma \subset \sigma_{\alpha_j} : \ \sigma \cup \sigma_{\alpha_j - q_{\alpha_j}} \in \Delta_{I_j}\} \ .$$
- We let
$$(C^\bullet(j,\alpha), d_j) = (C^\bullet(\Delta_{j,\alpha}), d_j) \ .$$
- Let $\tau \subset [n]$ be such that $\sigma_{\beta_i} \subset \tau\sigma_{\beta_j}$. We define $f(\tau, j, i) = \sigma_{\beta_j}\tau\widehat{\sigma_{\beta_i}}$.

- Let $\tau^* \in C^r(j,\alpha)$. Let $w \in [n]$ be such that $\tau w \notin \Delta_{j,\alpha}$. The coefficient $r_{j,i,\underline{w\tau}+\alpha_j-q_{\alpha_j}}$ is determined by Corollary 2.4. Whenever $r_{j,i,\underline{w\tau}+\alpha_j-q_{\alpha_j}} \neq 0$ it follows that $\sigma_{\beta_i} \subset \tau\sigma_{\beta_j}$ and $f(\tau,j,i) \in \Delta_{i,\alpha}$ so that
$$\chi_j(\tau^*, w) = \operatorname{sgn}[w, \tau w]\ r_{j,i,\underline{w\tau}+\alpha_j-q_{\alpha_j}}\ \frac{\operatorname{sgn}[\tau w, \sigma_{\alpha_j}]}{\operatorname{sgn}[f(\tau w, j, i), \sigma_{\alpha_i}]}\ (f(\tau w, j, i))^*$$
is an an element of
$$\sum_{i<j} C^{r+1+(|\sigma_{\alpha_i}|-|\sigma_{\alpha_j}|)}(i,\alpha).$$
We let
$$\chi_j(\tau^*) = \sum_{w \in V_{\tau,\Delta_{j,\alpha},2}} \chi_j(\tau^*, w)\ .$$

Example 4.2. Let A be the matrix of Example 3.2. Let $\alpha = (1,0,1,1)$. Then $\alpha_1 = (1,0,1,0)$, $\alpha_2 = (0,0,1,1)$, $\sigma_{\alpha_1} = \{1,3\}$, $\sigma_{\alpha_2} = \{3,4\}$ while $\sigma_{\beta_1} = \{4\}$ and $\sigma_{\beta_2} = \{1\}$. $\Delta_{1,\alpha}$ is the line segment between the vertices 1 and 3 while $\Delta_{2,\alpha}$ consists of the points 3 and 4. It follows that $f(\{3,4\},2,1) = \{1,3\}$, an element of $\Delta_{1,\alpha}$ and $r_{2,1,\{3,4\}} = -1/2$. Thus $\chi_2(\{3\}^*) = -\frac{1}{2}\{1,3\}^*$.

Next we turn our attention to the minimal multigraded free resolutions of M_A. Let $\alpha \in \mathbb{Z}^n$ and $b_{i,\alpha}(M_A)$ be the α-graded i-betti number of M_A:
$$b_{i,\alpha}(M_A) = \dim_{\Bbbk} \operatorname{Tor}_i(M_A, k)_\alpha = \dim_{\Bbbk} H_i(M_A \otimes K_\bullet)_\alpha = \dim_{\Bbbk} F_i \otimes \Bbbk$$
where $F_\bullet:\quad 0 \longrightarrow F_p \longrightarrow \cdots\cdots \longrightarrow F_1 \xrightarrow{\phi_1} F_0 \longrightarrow M_A \longrightarrow 0$ is a minimal multigraded free resolution of M_A, $(\phi_1 = \phi_T)$. It is well known that when I is a squarefree ideal then $b_{i,\alpha}(R/I) = \widetilde{H}^{|\sigma_\alpha|-i-1}(C^\bullet((\Delta_I)_\alpha))$, see [Ho77] or [MiSt05] for a proof. More precisely there is an isomorphism of complexes:
$$(R/I \otimes K_\bullet)_\alpha \cong C^\bullet((\Delta_I)_\alpha), \tag{4.3}$$
such that
$$(R/I \otimes K_i)_\alpha \cong C^{|\sigma_\alpha|-i-1}((\Delta_I)_\alpha).$$
We generalize the isomorphism (4.3) for M_A. We will need the following remark on the signs, proved essentially in [Ho77].

Remark 4.4. Let $\rho \subset \sigma$, $\tau = \sigma \setminus \rho$ and $t \in \tau$. Then
$$\operatorname{sgn}[t,\tau] \operatorname{sgn}[\rho t, \sigma] = \operatorname{sgn}[t, \rho t] \operatorname{sgn}[\rho, \sigma]\ .$$

We combine the cochain complexes of $\Delta_{j,\alpha}$ to construct a new complex.

Construction 4.5. *Let $l_j = |\sigma_{\alpha_j}| - |\sigma_{\alpha_1}|$. We define*
$$C^t(A,\alpha) := \sum_{j=1}^{s} C^{t+l_j}(j,\alpha)\ ,$$
and let $\delta^t:\ C^t(A,\alpha) \longrightarrow C^{t+1}(A,\alpha)$ be such that for $\tau^* \in C^{t+l_j}(j,\alpha)$,
$$\delta^t(\tau^*) := d_j^t(\tau^*) + \chi_j(\tau^*)\ .$$

Theorem 4.6. *Let A be a squarefree multigraded matrix. $(C^\bullet(A,\alpha), \delta^\bullet)$ is a cochain complex and there is an isomorphism of complexes $(C^\bullet(A,\alpha), \delta^\bullet) \cong (A \otimes K_\bullet)_\alpha$.*

Proof. First we note that $M_A \otimes K_\bullet$ is multigraded. By Corollary 2.4, a multigraded basis for the vector space $(M_A \otimes K_\bullet)_\alpha$ is

$$\bigcup_{j} \bigcup_{\substack{L \subset [n] \\ \alpha_j - \deg L \in \mathbb{N}^n}} \{\overline{x^{\alpha_j - \deg L}\ v_j} \otimes\ e_L :\ \overline{x^{\alpha_j - \deg L}\ v_j}\ \in B(M_A)\}$$

$$= \bigcup_{\substack{L \\ \alpha_j - \deg L \in \mathbb{N}^n}} \bigcup_{j} \{\overline{x^{\alpha_j - q_{\alpha_j}} x^{q_{\alpha_j} - \deg L} v_j} \otimes\ e_L :\ \sigma_{\alpha_j} \hat{L} \in \Delta_{j,\alpha}\}\ .$$

To each element $\overline{x^{\alpha_j - q_{\alpha_j}} x^{q_{\alpha_j} - \deg L} v_j} \otimes\ e_L$ of this basis we correspond the element $\operatorname{sgn}[\sigma_{\alpha_i} \backslash L, \sigma_{\alpha_i}]\ (\sigma_{\alpha_i} \backslash L)^*$ of $C^r(j, \alpha)$ where $r = |\sigma_{\alpha_j}| - |L| - 1$. Since $(M_A \otimes K_\bullet)_\alpha$ is a complex to prove our claim it suffices to show that the following diagram commutes:

$$\begin{array}{ccc} C^t(A, \alpha) & \xrightarrow{\delta^t} & C^{t+1}(A, \alpha) \\ \downarrow & & \downarrow \\ (M_A \otimes K_{|\sigma_{\alpha_1}| - t - 1})_\alpha & \xrightarrow{1_{M_A} \otimes \theta} & (M_A \otimes K_{|\sigma_{a_1}| - t})_\alpha . \end{array}$$

This is a routine check, using Remark 4.4. □

The following is now immediate and generalizes the well known formula of the cyclic case.

Corollary 4.7. *Let A be as above. Then*

$$b_{i,\alpha}(M_A) = H^{|\sigma_{\alpha_1}| - i - 1}(C^\bullet(A, \alpha)).$$

When $E^\bullet$ is a complex, by $E^\bullet[-1]$ we mean the complex $E^\bullet$ pushed in homological degree -1: $E^r[-1] := C^{r-1}$. This way we can think of $C^\bullet(A, \alpha)$ as the cochain complex that results by a succession of mapping cones $\mathbb{M}(f_i)$. We start with $\mathbb{M}(f_1) = C^\bullet(1, \alpha)[-1]$ and once $\mathbb{M}(f_{i-1})$ has been constructed then $\mathbb{M}(f_i)$ is the cokernel of

$$f_i : (C^\bullet(i, \alpha)[-|\sigma_{\alpha_i}| + |\sigma_{a_1}|], -d) \xrightarrow{d_i'} \mathbb{M}(f_{i-1})[-1].$$

We note that if $\alpha \in \mathbf{N}^n$ is not squarefree then for each i, $\Delta_{i,\alpha}$ is a cone and the cohomology of $C^\bullet(i, \alpha)$ is everywhere zero. It follows that the minimal resolution of M is supported in squarefree degrees, see [BrHe95], [Ya00].

Example 4.8. Let A be the matrix of Example 3.2 and let $\alpha = (1, 0, 1, 1)$. Then $l_1 = l_2 = 0$,

$$C^\bullet(1, \alpha) : \quad 0 \longrightarrow \Bbbk \longrightarrow \Bbbk^2 \longrightarrow \Bbbk \longrightarrow 0$$

$$C^\bullet(2, \alpha) : \quad 0 \longrightarrow \Bbbk \longrightarrow \Bbbk^2 \longrightarrow 0$$

and

$$C^\bullet(A, \alpha) : \quad 0 \longrightarrow \Bbbk^2 \longrightarrow \Bbbk^4 \longrightarrow \Bbbk \longrightarrow 0\ .$$

It follows that $\dim_\Bbbk H^0(C^\bullet(A, \alpha)) = 1$ and $b_{1,\alpha}(M_A) = 1$ as expected.

5. Local cohomology of M

Let $A = (c_{ij}x^{\alpha_{ij}})$ be an $s \times l$ multigraded squarefree matrix, T a squarefree solution of E_A, $\phi = \phi_T$, $M_A = \operatorname{Coker}\phi$ and we let $I_1, \dots, I_s$ be the squarefree monomial ideals as in the previous section. We proceed with the notation and related remarks.

Notation 5.1.

- Let $F \subset [n]$. Let H be any R-module. We write H_F for the localization of H at the powers of $x^{\underline{F}}$. In particular for $F = \{i_1, \dots, i_t\}$, $R_F = \Bbbk[x_1, \dots, x_n, x_{i_1}^{-1}, \dots, x_{i_t}^{-1}]$. Let $u \in H$. We write u_F to denote the image of u in H_F under the natural homomorphism $H \longrightarrow H_F$. If $\phi : H_1 \longrightarrow H_2$ is an R-homomorphism we write $\phi_F : (H_1)_F \longrightarrow (H_2)_F$ for the induced homomorphism.
- We let $A_F = (c_{ij}x_F^{\alpha_{ij}})$. We recall that A is the matrix of $\phi :\ F_1 \longrightarrow F_0$ with respect to bases $\{w_i :\ i = 1, \dots, l\}$ of F_1 and $\{v_j :\ j = 1, \dots, s\}$ of F_0. Thus A_F is the matrix of $\phi_F :\ (F_1)_F \longrightarrow (F_0)_F$ with respect to the bases $\{(w_i)_F :\ i = 1, \dots, l\}$ of $(F_1)_F$ and $\{(v_j)_F :\ j = 1, \dots, s\}$ of $(F_0)_F$ and the multidegrees of $(w_i)_F$, $(v_j)_F$, $i \in [l]$, $j \in [s]$ are squarefree. For each j we let $I_{j,F} = (I_j)_F$.
- Let Δ be a simplicial complex on $[n]$ and let $\alpha \in \mathbb{Z}^n$. We let
$$\Delta^\alpha = \{\tau :\ \tau \cap \sigma_{a^-} = \emptyset,\ \tau \cup \sigma_\alpha \in \Delta\}\ .$$
We note that if $\alpha = -\alpha^-$ and $\sigma_{\alpha^+} = \emptyset$ then Δ^α is by definition the link of σ_α in Δ.
- Let $F, G \subset [n]$. Let N be an R-module. We let $\theta_{F,G} :\ M_F \longrightarrow M_G$, $\theta_{F,G}(u_F) = u_G$ if $G = Fh$ and zero otherwise. We let $K(x^\infty, N)$ to be the complex
$$K(x^\infty, N) :\ 0 \to N \xrightarrow{\theta^0} \bigoplus_{\substack{|F|=1 \\ F \subset [n]}} N_F \xrightarrow{\theta^1} \cdots \to N_{[n]} \to 0$$
where $\theta^r|_{N_F} = (\theta_{F,G})$. It is well known that for any multigraded module N and $\alpha \in \mathbb{Z}^n$,
$$H_m^i(N)_\alpha = H^i(K(x^\infty, N)_\alpha)\ ,$$
see [BrHe98]. Moreover when I is a squarefree monomial ideal, then by reordering the variables of R so that the indices of σ_{α^-} are at the end of $[n]$ one gets:
$$K(x^\infty, R/I)_\alpha \cong C^\bullet((\Delta_I)^\alpha)[-|\sigma_{\alpha^-}| - 1] \tag{5.2}$$
and
$$\dim_\Bbbk H_m^i(R/I)_\alpha = \dim_\Bbbk H^{i-|\sigma_{\alpha^-}|-1}(C^\bullet((\Delta_I)^\alpha))\ ,$$
see [St83] or [BrHe98].
- We recall from the previous section that if $\alpha \in \mathbf{Z}^n$ then $\alpha_i = \alpha - \beta_i$. We denote by Δ_j^α the complex $(\Delta_{I_j})^{\alpha_j}$. Thus
$$\Delta_j^\alpha = \{\tau :\ \tau \cap \sigma_{a_{j^-}} = \emptyset,\ \tau \cup \sigma_{\alpha_j} \in \Delta_{I_j}\}\ .$$

If $F \subset [n]$ we let $B_{j,F} := \{\overline{x^\gamma v_{j,F}} :\ x^\gamma \in R_F,\ x^\gamma \notin I_{j,F}\}$. We let

$$B_F(A) = \bigcup_j B_{j,F}\ .$$

We note that $B_{j,F} = \{\overline{x^\gamma v_{j,F}} :\ \sigma_{\gamma^-} \subset F,\ F \cup \sigma_{\gamma^+} \in \Delta_{I_j}\}$. Moreover $\deg(\overline{x^\gamma v_{j,F}}) = \gamma + \beta_j$. Thus the elements of $B_F(A)$ of degree α form the set $B_F(A)_\alpha = B_F(A) \cap \{\overline{x^\gamma v_{i,F}} :\ \gamma + \beta_i = \alpha,\ i = 1, \ldots, s\} = \{\overline{x^{\alpha_i} v_{iF}} :\ \sigma_{\alpha_i^-} \subset F,\ F\widehat{\sigma_{\alpha_i^-}} \in \Delta_i^\alpha,\ i = 1, \ldots, s\}$. In the next Theorem we determine a $\Bbbk$-basis for $K(x^\infty, M_A)^r_\alpha$ and its various homological components.

Theorem 5.3. *Let A be as above. Then*

(1) *$B_F(A)$ is a multigraded $\Bbbk$-basis for $(M_A)_F$. Moreover if $x^\gamma \in I_{i,F}$ then*

$$\overline{x^\gamma v_{i,F}} = \sum_{j<i} r_{i,j,\gamma^+ + \underline{F}}\ \overline{x^{\gamma+\beta_i-\beta_j} v_{j,F}}\ .$$

(2) *Let $\alpha \in \mathbb{Z}^n$. The set*

$$B_\alpha(A) = \bigcup_{F \subset [n]} B_F(A)_\alpha$$

is a $\Bbbk$-basis for $K(x^\infty, M_A)_\alpha$.

(3) *The set*

$$B_{\alpha,r}(A) = \{\overline{x^{\alpha_i} v_{iF}} : F \subset [n], |F| = r,\ \sigma_{\alpha_i^-} \subset F,\ F\widehat{\sigma_{\alpha_i^-}} \in \Delta_i^\alpha\}$$

is a $\Bbbk$-basis for $K(x^\infty, M_A)^r_\alpha$.

Proof. We prove the first claim. The rest follows by degree consideration. First we prove linear independence. Suppose that $x^\gamma \notin I_{j,F}$. Let $\beta \in \mathbb{N}^n$ such that $\sigma_{\gamma^-} \subset \sigma_\beta \subset F$. Then $x^\beta x^\gamma \notin I_j$. Thus by clearing denominators, any possible linear dependence relation on the elements of $B_F(A)$ corresponds to a linear dependence relation on the elements of $B(M_A)$.

Next we show that $B_F(A)$ spans $(M_A)_F$. Let $\gamma \in \mathbb{N}^n$, such that x^γ is a generator of $I_{i,F}$. Then $x^{\gamma^+} x^{\underline{F}}$ is a generator of I_i. By Corollary 2.4 it follows that

$$\overline{x^{\gamma^+ + \underline{F}}\ v_i} = \sum_{j<i} r_{i,j,\gamma^+ + \underline{F}}\ \overline{x^{\gamma^+ + \beta_i - \beta_j}\ v_j}\ .$$

Localizing at the powers of $x^{\underline{F}}$ and dividing by $x^{\gamma^-} x^{\underline{F}}$ we get the desired claim. $\square$

Next we describe the complex that will be used to compute $H^i_m(M_A)_\alpha$. First we need to define one more sign: let $\sigma \subset F$. We reorder F so that the elements of σ are at the end of F. If the number of transpositions needed to do this is even we let $t(\sigma, F) := 1$, otherwise we let $t(\sigma, F) := -1$. If $h \notin F$, it is direct to verify that

$$t(\sigma, F)\,\mathrm{sgn}[h, Fh\hat{\sigma}] = t(\sigma, Fh)\,\mathrm{sgn}[h, Fh]\ .$$

Theorem 5.4. *Let A be as above and $\alpha \in \mathbb{Z}^n$. Let $(L^\alpha_j)^\bullet = C^\bullet(\Delta^a_j)$ and $l^-_j = |\sigma_{a^-_i}| - |\sigma_{a^-_1}|$. For each $r \in \mathbb{Z}$ we let*

$$(L^\alpha)^r := \sum_{i=1}^r L_i^{r-l_i^-}$$

and $d^r : (L^\alpha)^r \longrightarrow (L^\alpha)^{r+1}$ *be such that when* $\tau \in \Delta_i^\alpha$, $|\tau| = r + 1 - l_i^-$ *then*

$$\tau^* \mapsto \sum_{h \in V_{\tau,\Delta_i^\alpha,1}} \mathrm{sgn}[h, \tau h]\ (\tau h)^*$$

$$+ \sum_{h \in V_{\tau,\Delta_i^\alpha,2}} \mathrm{sgn}[h, \tau h]\ \sum_{j<i} r_{i,j,\alpha_i^- + \underline{\tau h}} \frac{t(\sigma_{\alpha_i^-}, \tau h\ \sigma_{\alpha_i^-})}{t(\sigma_{\alpha_j^-}, \tau h \sigma_{\alpha_i^-})}, (\tau h\ \sigma_{\alpha_i^-}\ \widehat{\sigma_{\alpha_j^-}})^* \ .$$

$((L^\alpha(A))^\bullet, d^\bullet)$ *is a cochain complex and*

$$\dim_{\Bbbk} H_m^i(M_A)_\alpha = \dim_{\Bbbk} H^{i-|\sigma_{\alpha_1^-}|-1}((L^\alpha(A))^\bullet).$$

Proof. There is an isomorphism of vector spaces

$$K(x^\infty, M_A)_\alpha^r \cong \sum_i (L_i^\alpha)^{r-|\sigma_{a_i^-}|-1}$$

where

$$\overline{x^{\alpha_i} v_{i,F}} \mapsto (F\widehat{\sigma_{a_i^-}})^* \ .$$

It is routine to show that the following diagram commutes:

$$\begin{array}{ccc} (L^a)^t & \xrightarrow{d^t} & (L^a)^{t+1} \\ \downarrow & & \downarrow \\ K(x^\infty, M_A)_\alpha^{t+|\sigma_{a_1^-}|+1} & \longrightarrow & K(x^\infty, M_A)_\alpha^{t+|\sigma_{a_1^-}|+2} \end{array} .$$

□

Example 5.5. Let A be the matrix of Example 3.2 and let $\alpha = (0, -1, -1, 0)$. Then $\alpha_1 = (0, -1, -1, -1)$, $\alpha_2 = (-1, -1, -1, 0)$, $l_1^- = l_2^- = 0$. Moreover $\Delta_1^\alpha = \Delta_2^\alpha = \{\emptyset\}$,

$$(L^\alpha)^\bullet : \quad 0 \longrightarrow \Bbbk^2 \longrightarrow 0 \ ,$$

and $\dim_{\Bbbk} H^{-1}(L^\alpha) = 2$. It follows that $\dim_{\Bbbk} H_m^3(M_A)_\alpha = 2$. We do in more detail the case for $\alpha = (0, 0, 0, 0)$. Here $\alpha_1 = (0, 0, 0, -1)$, $\alpha_2 = (-1, 0, 0, 0)$, $l_1^- = l_2^- = 0$, $\sigma_{\alpha_1^-} = \{4\}$ and $\sigma_{\alpha_2^-} = \{1'\}$. Δ_1^α has facets the boundary of the triangle $\{1, 2, 3\}$ while the facets of Δ_2^α are $\{4\}$ and $\{2, 3\}$. We have

$$L^\alpha : \ 0 \longrightarrow \Bbbk^2 \longrightarrow \Bbbk^6 \longrightarrow \Bbbk^4 \longrightarrow 0$$

with zero cohomology at all homological degrees. For $\tau = \{2\} \in \Delta_2^\alpha$, $V_{\tau,\Delta_2^\alpha,1} = \{3\}$, $V_{\tau,\Delta_2^\alpha,2} = \{4\}$ and $d^0(\tau^*) = \tau_1 - \tau_2$ where $\tau_2 = \{2, 3\}$ (in Δ_2^α) and $\tau_1 = \{1, 2\}$ (in Δ_1^α).

We finish this section with a corollary whose proof is immediate.

Corollary 5.6. *Let* A *be as above. If for some* $\alpha \in \mathbb{Z}^n$, $H_m^i(M)_\alpha \neq 0$, *then* $\dim_{\Bbbk} H_m^i(M_A)_\beta = \dim_{\Bbbk} H_m^i(M_A)_\alpha$ *for all* $\beta \in \mathbb{Z}^n$ *such that* $\sigma_{\beta^+} = \sigma_{\alpha^+}$ *and* $\sigma_{\beta^-} = \sigma_{\alpha^-}$.

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Department of Mathematics, Aristotle University of Greece, Thessaloniki, 54124
E-mail address: `hara@math.auth.gr`